

# REAL-ROOT PROPERTY OF THE SPECTRAL POLYNOMIAL OF THE TREIBICH-VERDIER POTENTIAL AND RELATED PROBLEMS

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**ABSTRACT.** We study the spectral polynomial of the Treibich-Verdier potential. Such spectral polynomial, which is a generalization of the classical Lamé polynomial, plays fundamental roles in both the finite-gap theory and the ODE theory of Heun's equation. In this paper, we prove that all the roots of such spectral polynomial are real and distinct under some assumptions. The proof uses the classical concept of Sturm sequence and isomonodromic theories. We also prove an analogous result for a polynomial associated with a generalized Lamé equation. Differently, our new approach is based on the viewpoint of the monodromy data.

## 1. INTRODUCTION

Throughout the paper, we use the notations  $\omega_0 = 0$ ,  $\omega_1 = 1$ ,  $\omega_2 = \tau$ ,  $\omega_3 = 1 + \tau$  and  $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$ , where  $\tau \in \mathbb{H} = \{\tau \mid \text{Im } \tau > 0\}$ . Define  $E_\tau := \mathbb{C}/\Lambda_\tau$  to be a flat torus in the plane and  $E_\tau[2] := \{\frac{\omega_k}{2} \mid 0 \leq k \leq 3\} + \Lambda_\tau$  to be the set consisting of the lattice points and 2-torsion points in  $E_\tau$ .

In the literature, a smooth period function  $q(z)$  satisfying  $q(z) \in \mathbb{R}$  for  $z \in \mathbb{R}$  is called a *finite-gap potential* if the set  $\sigma_b(H)$  of  $H = -d^2/dz^2 + q(z)$  satisfies

$$\overline{\sigma_b(H)} \cap \mathbb{R} = [E_0, E_1] \cup [E_2, E_3] \cup \cdots \cup [E_{2g}, +\infty)$$

with  $E_0 < E_1 < \cdots < E_{2g}$ , where  $\sigma_b(H)$  is the spectrum of bounded bands, that is,

$$E \in \sigma_b(H) \Leftrightarrow \text{Every solution of } (H - E)f(z) = 0 \text{ is bounded on } z \in \mathbb{R}.$$

Recall that  $\wp(z) = \wp(z|\tau)$  is the Weierstrass elliptic function with periods 1 and  $\tau$ , defined by

$$\wp(z|\tau) := \frac{1}{z^2} + \sum_{\omega \in \Lambda_\tau \setminus \{0\}} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right),$$

and  $e_k = e_k(\tau) := \wp(\frac{\omega_k}{2}|\tau)$  for  $k \in \{1, 2, 3\}$ . In [13], Ince proved that if  $\tau \in i\mathbb{R}_{>0}$  (i.e.  $E_\tau$  is a rectangle torus) and  $n \in \mathbb{N}$ , then the potential  $n(n+1)\wp(z + \frac{\omega_k}{2}|\tau)$ ,  $k \in \{2, 3\}$ , is a finite-gap potential. Surprisingly, the finite-gap potential is related to the KdV theory. A potential  $q(z)$  is called an *algebro-geometric finite-gap potential* if there is an odd-order differential operator  $A = (d/dz)^{2m+1} + \sum_{j=0}^{2m-1} b_j(z)(d/dz)^{2m-1-j}$  such that

$[A, -d^2/dz^2 + q(z)] = 0$ , that is,  $q(z)$  is a solution of KdV hierarchy equations (cf. [2]). Under the condition that  $q(z)$  is smooth periodic and real-valued on  $\mathbb{R}$ , it is known (cf. [10]) that  $q(z)$  is a finite-gap potential if and only if it is an algebro-geometric finite-gap potential.

In 1990's, Treibich and Verdier found a new algebro-geometric finite-gap potential, which is now called the *Treibich-Verdier potential* ([22]). This potential could be written as

$$(1.1) \quad q^{(l_0, l_1, l_2, l_3)}(z) := \sum_{k=0}^3 l_k(l_k + 1) \wp\left(z + \frac{\omega_k}{2}\right),$$

where  $l_k \in \mathbb{N} \cup \{0\}$  for all  $k$ . See [14, 17, 18, 19, 20, 21] and references therein for historical reviews and subsequent developments. Notice that  $E_j$  for the potential (1.1) might not be real in general; see Remark 3.3 below.

The polynomial  $\prod_{i=0}^{2g} (E - E_i)$  is called the *spectral polynomial* of the operator  $H = -d^2/dz^2 + q(z)$ . A basic question is how to determine the spectral polynomial of the operator

$$(1.2) \quad H^{(l_0, l_1, l_2, l_3)} := -\frac{d^2}{dz^2} + \sum_{k=0}^3 l_k(l_k + 1) \wp\left(z + \frac{\omega_k}{2}\right).$$

Remark that  $H^{(l_0, l_1, l_2, l_3)}$  is also the Hamiltonian of the  $BC_1$  (one particle) Inozemtsev model (cf. [17]).

In the literature, there are two methods to compute the spectral polynomials. One way is to study the so-called "polynomial solutions" of the following generalized Lamé equation (GLE):

$$(1.3) \quad y''(z) = I(z)y(z) = \left[ \sum_{k=0}^3 l_k(l_k + 1) \wp\left(z + \frac{\omega_k}{2}\right) - E \right] y(z) \text{ in } E_\tau,$$

where  $E \in \mathbb{C}$ . GLE (1.3) is a Fuchsian equation with singular points in  $E_\tau$  [2]. By projecting  $E_\tau$  onto  $\mathbb{CP}^1$  via  $x = \wp(z)$ , GLE (1.3) becomes a second order Fuchsian equation with four singular points  $\{e_1, e_2, e_3, \infty\}$  on  $\mathbb{CP}^1$ . See (3.3) in Section 3 for this new ODE. Therefore, GLE (1.3) is an elliptic form of Heun's equation. This fact was first pointed out by Darboux [9] in 1882. Classically, people are interested in finding the condition on  $E$  such that the new ODE (3.3) has a "polynomial" solution in terms of  $x$  (see Section 3 for a precise definition). It is known that there is a *polynomial*

$$Q(E) = Q^{(l_0, l_1, l_2, l_3)}(E) = Q^{(l_0, l_1, l_2, l_3)}(E|\tau)$$

such that the new ODE (3.3) has a "polynomial" solution if and only if  $Q(E) = 0$ . Furthermore, it was proved in [21, Section 6] that

$$(1.4) \quad \text{this polynomial } Q^{(l_0, l_1, l_2, l_3)}(E|\tau) \text{ coincides with the spectral polynomial of the operator } H^{(l_0, l_1, l_2, l_3)} \text{ up to a multiplication.}$$

Therefore in this paper, we also denote by  $Q^{(l_0, l_1, l_2, l_3)}(E|\tau)$  to be the *spectral polynomial* of the operator  $H^{(l_0, l_1, l_2, l_3)}$ .

When  $(l_0, l_1, l_2, l_3) = (n, 0, 0, 0)$ , GLE (1.3) turns to be the *classical Lamé equation* ([12, 15, 23])

$$(1.5) \quad y''(z) = [n(n+1)\wp(z) + B]y(z) \text{ in } E_\tau,$$

and the corresponding polynomial  $\ell_n(B|\tau) := Q^{(n, 0, 0, 0)}(-B|\tau)$  is called the *Lamé polynomial* in the literature; see e.g. [23, 15]. It is known that  $\deg_B \ell_n = 2n + 1$ . We refer to [1, 7] for the general theory of the Lamé equation and explicit forms of the Lamé polynomial.

A remarkable result about  $\ell_n(B|\tau)$  which is related to the finite-gap phenomena is the following *real-root property*.

**Theorem A.** (cf. [23]) *Let  $\tau \in i\mathbb{R}_{>0}$  and  $n \in \mathbb{N}$ . Then all the roots of  $\ell_n(\cdot|\tau)$  are real and distinct.*

Theorem A has another important application. We associate the polynomial  $\ell_n(B|\tau)$  with a *hyperelliptic curve*  $Y_n(\tau) := \{(B, W) | W^2 = \ell_n(B|\tau)\}$ . Since  $\ell_n(B|\tau) \in \mathbb{Q}[g_2(\tau), g_3(\tau)][B]$ , where  $g_2, g_3$  are coefficients of

$$\wp'(z|\tau)^2 = 4\wp(z|\tau)^3 - g_2(\tau)\wp(z|\tau) - g_3(\tau),$$

Theorem A implies that the discriminant of  $\ell_n(\cdot|\tau)$ , which defines a modular form with respect to  $SL(2, \mathbb{Z})$ , has only finitely many zeros modulo  $SL(2, \mathbb{Z})$ . This means that *except for finitely many tori,  $\ell_n(B|\tau)$  has no multiple roots, or in other words, the associated hyperelliptic curve  $Y_n(\tau)$  is smooth at its finite branch points*. See [7, Theorem 7.4]. This hyperelliptic curve has some interesting geometric properties. For example, it was shown in [7] that  $Y_n(\tau) \cup \{\infty\}$  is a cover of the torus  $E_\tau$ .

The above argument highlights the importance of studying the following question:

**(Q):** *For  $l_0, l_1, l_2, l_3 \in \mathbb{N} \cup \{0\}$  and  $\tau \in i\mathbb{R}_{>0}$ , whether are all the roots of  $Q^{(l_0, l_1, l_2, l_3)}(E|\tau)$  real and distinct?*

Theorem A already solves the special case  $l_1 = l_2 = l_3 = 0$ . Later, this question was partially answered by the forth author [17].

**Theorem B.** [17, Proposition 3.3] *Let  $\tau \in i\mathbb{R}_{>0}$ ,  $l_2 = l_3 = 0$  and  $l_0, l_1 \in \mathbb{N} \cup \{0\}$ . Then all the roots of  $Q^{(l_0, l_1, l_2, l_3)}(\cdot|\tau)$  are real and distinct.*

One main purpose of this paper is to settle question (Q) in new cases. Our first main result is following.

**Theorem 1.1.** *Let  $\tau \in i\mathbb{R}_{>0}$  and  $l_0, l_1, l_2, l_3 \in \mathbb{N} \cup \{0\}$ . Then all the roots of  $Q^{(l_0, l_1, l_2, l_3)}(\cdot|\tau)$  are real and distinct provided that one of the following conditions holds.*

- (i)  $l_3 = 0$  and  $l_0 \geq l_1 + l_2 - 1$ .
- (ii)  $l_0 + l_3 + 1 = l_1 + l_2$ ,  $l_2 + l_3 \geq l_0 + l_1 + 1$ ,  $l_1 + l_3 \geq l_0 + l_2 + 1$ .
- (iii)  $l_0 + l_3 = l_1 + l_2 + 1$ ,  $l_0 + l_1 \geq l_2 + l_3 + 1$ ,  $l_0 + l_2 \geq l_1 + l_3 + 1$ .

We note that (i) is a generalization of Theorem B. In the case  $l_3 = 0$ , Remark 3.3 indicates that the condition  $l_0 \geq l_1 + l_2 - 1$  is sharp in the sense that  $Q^{(2,2,0)}(\cdot|\tau)$  has two non-real roots even for  $\tau \in i\mathbb{R}_{>0}$ . Our proof of Theorem 1.1 will apply the classical concept of Sturm sequence and some isomonodromic results which were obtained via generalized Darboux transformations in [21]. See Sections 2-3.

The second method to compute the polynomial  $Q$  is to use the second symmetric product of GLE (1.3). This method is also well-known and closely related to the monodromy representation of GLE (1.3). See e.g. [23, 17, 7]. Indeed, this method could work for a class of ODEs including GLE (1.3) and, particularly, its generalization:

$$(1.6) \quad y''(z) = I^{(l_0, l_1, l_2, l_3)}(z; p, \tau) y(z) \text{ in } E_\tau,$$

$$\begin{aligned} \text{with } I^{(l_0, l_1, l_2, l_3)}(z; p, \tau) = & q^{(l_0, l_1, l_2, l_3)}(z) + \frac{3}{4}(\wp(z+p) + \wp(z-p)) \\ & + A(\zeta(z+p) - \zeta(z-p)) + B, \end{aligned}$$

where  $q^{(l_0, l_1, l_2, l_3)}$  is in (1.1),  $A, B \in \mathbb{C}$  and  $\pm p \notin E_\tau[2]$  are always assumed to be *apparent singularities* (i.e. non-logarithmic). Under this assumption,  $B$  is determined by  $(p, A)$  as follows (see [3]):

$$(1.7) \quad B = A^2 - \zeta(2p)A - \frac{3}{4}\wp(2p) - \sum_{k=0}^3 l_k(l_k + 1)\wp\left(p + \frac{\omega_k}{2}\right).$$

Here we recall that  $\zeta(z) = \zeta(z|\tau) := -\int^z \wp(\xi|\tau)d\xi$  is the Weierstrass zeta function with two quasi-periods:

$$(1.8) \quad \eta_1(\tau) = \zeta(z+1|\tau) - \zeta(z|\tau), \quad \eta_2(\tau) = \zeta(z+\tau|\tau) - \zeta(z|\tau).$$

We are interested in GLE (1.6) because it can be reduced to (1.3) with different values of  $l_k$  by letting  $p \rightarrow \frac{\omega_k}{2}$  and has also a close relation to the well-known Painlevé VI equation. For example, if  $(A(\tau), B(\tau), p(\tau))$  depends on  $\tau$  suitably such that GLE (1.6) preserves the monodromy as  $\tau$  deforms, then  $p(\tau)$  satisfies the elliptic form of Painlevé VI equation. See [3].

By applying the second method to GLE (1.6), we can obtain a hyperelliptic curve  $Y_p^{(l_0, l_1, l_2, l_3)}(\tau) = \{(A, W) | W^2 = Q^{(l_0, l_1, l_2, l_3)}(A; p, \tau)\}$  associated with GLE (1.6), where

$$Q^{(l_0, l_1, l_2, l_3)}(A; p, \tau) \in \mathbb{Q}[\wp(p|\tau), \wp'(p|\tau), e_1(\tau), e_2(\tau), e_3(\tau)][A]$$

is a polynomial of  $A$ . Therefore,  $Q^{(l_0, l_1, l_2, l_3)}(A; p, \tau) \in \mathbb{R}[A]$  if  $\tau \in i\mathbb{R}_{>0}$  and  $p \in (0, \frac{1}{2})$ . Similarly as  $Y_n(\tau)$ ,  $Y_p^{(l_0, l_1, l_2, l_3)}(\tau) \cup \{\infty\}$  has a natural covering over  $E_\tau$  (see [6]). Naturally we ask the following question: for fixed  $p \in (0, \frac{1}{2})$ , is  $Y_p^{(l_0, l_1, l_2, l_3)}(\tau)$  smooth except for finitely many tori, or equivalently, does the polynomial  $Q^{(l_0, l_1, l_2, l_3)}(A; p, \tau)$  have distinct roots except for finitely many tori?

This question seems not trivial because the form of  $\mathcal{Q}^{(l_0, l_1, l_2, l_3)}(A; p, \tau)$  is very complicated even for small  $l_k$ . For  $(l_0, l_1, l_2, l_3) = (1, 0, 0, 0)$ , we denote  $y = A\wp'(p)$ ,  $x = \wp(p)$  and write  $\hat{\ell}_1(y; x, \tau) = \mathcal{Q}^{(1, 0, 0, 0)}(A; p, \tau)$  for convenience. Then a calculation (see [6]) shows that  $\deg_y \hat{\ell}_1 = 6$  and

$$\begin{aligned} \hat{\ell}_1(y; x, \tau) = & [65536(4x^3 - g_2x - g_3)^3]^{-1} [262144y^6 - 196608(12x^2 - g_2)y^5 \\ & + 12288(400x^4 - 88g_2x^2 + g_2^2 - 64g_3x)y^4 + 2048(8000x^6 \\ & - 2512g_2x^4 + 380g_2^2x^2 - 7g_2^3 - 256g_3x^3 + 488g_2g_3x + 320g_3^2)y^3 \\ & - 192(12x^2 - g_2)(40000x^6 - 17680g_2x^4 + 2028g_2^2x^2 - 3g_2^3 \\ & - 15872g_3x^3 + 3712g_2g_3x + 1792g_3^2)y^2 + 16(9600000x^{10} \\ & - 7276800g_2x^8 + 1692032g_2^2x^6 - 134176g_2^3x^4 + 428g_2^4x^2 + 27g_2^5 \\ & - 7741440g_3x^7 + 3287040g_2g_3x^5 - 386560g_2^2g_3x^3 + 2944g_2^3g_3x \\ & + 1308672g_3^2x^4 - 316416g_2g_3^2x^2 + 896g_2^2g_3^2 - 98304g_3^3x)y \\ & - (8000x^6 - 3280g_2x^4 - 4g_2^2x^2 + 9g_2^3 - 4864g_3x^3 + 64g_2g_3x - 256g_3^2) \\ & (11200x^6 - 6896g_2x^4 + 916g_2^2x^2 + 3g_2^3 - 8192g_3x^3 + 2048g_2g_3x + 1024g_3^2)]. \end{aligned}$$

Our second result of this paper is following.

**Theorem 1.2.** *Let  $\tau \in i\mathbb{R}_{>0}$ . Then all the roots of  $\hat{\ell}_1(\cdot; x, \tau) = 0$  are real and distinct whenever  $x > e_1(\tau)$  or  $x < e_2(\tau)$ . In other words,*

- (1) *if  $\tau \in i\mathbb{R}_{>0}$  and  $p \in (0, \frac{1}{2})$ , then all the roots of  $\mathcal{Q}^{(1, 0, 0, 0)}(\cdot; p, \tau) = 0$  are real and distinct;*
- (2) *if  $\tau \in i\mathbb{R}_{>0}$  and  $p \in (0, \frac{\tau}{2})$ , then all the roots of  $\mathcal{Q}^{(1, 0, 0, 0)}(\cdot; p, \tau) = 0$  are purely imaginary and distinct.*

Here  $(0, \frac{\tau}{2}) := \{s\tau | s \in (0, \frac{1}{2})\}$ . The difference of the assertions (1) and (2) comes from the well known fact that for  $\tau \in i\mathbb{R}_{>0}$ ,  $\wp'(p|\tau) \in \mathbb{R}$  if  $p \in (0, \frac{1}{2})$  and  $\wp'(p|\tau) \in i\mathbb{R}$  if  $p \in (0, \frac{\tau}{2})$ .

Remark that due to the appearance of singularities  $\pm p \notin E_\tau[2]$ , Theorem 1.2 can not be proved via the same idea of Sturm sequence as Theorem 1.1. Our new approach of proving Theorem 1.2 contains two steps. The first step is to write down the equation for the (not completely reducible) monodromy data  $C$ 's of GLE (1.6) with  $(l_0, l_1, l_2, l_3) = (1, 0, 0, 0)$ , and the second one is to prove that such  $C$ 's are purely imaginary and distinct if  $\tau \in i\mathbb{R}_{>0}$ . Since the monodromy of GLE (1.6) is not completely reducible if and only if  $\mathcal{Q}^{(l_0, l_1, l_2, l_3)}(A; p, \tau) = 0$  (see [6]), the equation of the monodromy data  $C$ 's is also a polynomial of degree 6. This new polynomial has some advantages: (i) It can be decomposed as a product of four polynomials; (ii) It has a nice structure for each factors in (i). Therefore, we do not need to know the explicit formula of  $\hat{\ell}_1(y; x, \tau)$  in the proof of Theorem 1.2.

The paper is organized as follows. Theorem 1.1 will be proved in Sections 2-3 and Theorem 1.2 will be proved in Sections 4-5.

## 2. POLYNOMIAL SOLUTIONS AND STURM SEQUENCE

The purpose of this and next sections is to prove Theorem 1.1. To apply the idea of Sturm sequence, we need to investigate polynomial solutions of

$$(2.1) \quad \frac{d^2 y}{dx^2} + \left( \frac{\gamma_1}{x-t_1} + \frac{\gamma_2}{x-t_2} + \frac{\gamma_3}{x-t_3} \right) \frac{dy}{dx} + \frac{\alpha\beta(x-t_3)-q}{\prod_{j=1}^3(x-t_j)} y = 0.$$

It is a Fuchsian equation on  $\mathbb{CP}^1$  with four regular singularities  $\{t_1, t_2, t_3, \infty\}$ . We impose the condition  $\alpha + \beta + 1 = \gamma_1 + \gamma_2 + \gamma_3$  so that the exponents at  $x = \infty$  are  $\alpha$  and  $\beta$ . Set

$$(2.2) \quad y = \sum_{m=0}^{\infty} c_m(x-t_3)^m, \quad \text{where } c_0 = 1,$$

and substitute it to the differential equation which is multiplied by  $\prod_{j=1}^3(x-t_j)$  to equation (2.1). Then the coefficients satisfy the following recursive relations:

$$(2.3) \quad \begin{aligned} (t_1-t_3)(t_2-t_3)\gamma_3 c_1 &= qc_0, \\ (t_1-t_3)(t_2-t_3)(m+1)(m+\gamma_3)c_{m+1} &= -(m-1+\alpha)(m-1+\beta)c_{m-1} \\ &\quad + [m\{(m-1+\gamma_3)(t_1+t_2-2t_3)+(t_2-t_3)\gamma_1+(t_1-t_3)\gamma_2\}+q]c_m. \end{aligned}$$

If  $t_1 \neq t_2 \neq t_3 \neq t_1$  and  $\gamma_3 \notin -\mathbb{Z}_{\geq 0}$ , then it is easy to see that  $c_r$  is a polynomial in  $q$  of degree  $r$  and we denote it by  $c_r(q)$ .

Moreover we assume that  $\alpha = -N$  or  $\beta = -N$  for some  $N \in \mathbb{Z}_{\geq 0}$ . Let  $q_0$  be a solution to the equation  $c_{N+1}(q) = 0$ . Then it follows from (2.3) for  $m = N+1$  that  $c_{N+2}(q_0) = 0$ . By applying (2.3) for  $m = N+2, N+3, \dots$ , we have  $c_m(q_0) = 0$  for  $m \geq N+3$ . Hence, if  $c_{N+1}(q_0) = 0$ , then the differential equation (2.1) have a non-zero polynomial solution. More precisely, we obtain the following proposition.

**Proposition 2.1.** *Assume that  $t_1 \neq t_2 \neq t_3 \neq t_1$ ,  $\gamma_3 \notin -\mathbb{Z}_{\geq 0}$ ,  $\{\alpha, \beta\} = \{-N, \gamma_1 + \gamma_2 + \gamma_3 + N - 1\}$  and  $N \in \mathbb{Z}_{\geq 0}$ . If  $q$  is a solution to the equation  $c_{N+1}(q) = 0$ , then the differential equation (2.1) have a non-zero polynomial solution of degree no more than  $N$ .*

Next, we restrict to the case that all the parameters are *real*. The following proposition is shown immediately by applying the recursive equation (2.3).

**Proposition 2.2.** *Assume that  $\{\alpha, \beta\} = \{-N, \gamma_1 + \gamma_2 + \gamma_3 + N - 1\}$ ,  $N \in \mathbb{Z}_{\geq 0}$ ,  $\gamma_1, \gamma_2$  and  $\gamma_3$  are real,  $\gamma_3 > 0$ ,  $\gamma_1 + \gamma_2 + \gamma_3 + N > 1$  and  $(t_1-t_3)(t_2-t_3) < 0$ . Then the followings hold:*

- (i) *The sign of the coefficient of  $q^m$  in  $c_m(q)$  is that of  $(-1)^m$ .*
- (ii) *If  $1 \leq m \leq N$  and  $c_m(q) = 0$ , then the values of  $c_{m+1}(q)$  and  $c_{m-1}(q)$  are opposite in sign.*

By applying the argument of Sturm sequence, we have:

**Theorem 2.3.** *Assume that  $\{\alpha, \beta\} = \{-N, \gamma_1 + \gamma_2 + \gamma_3 + N - 1\}$ ,  $N \in \mathbb{Z}_{\geq 0}$ ,  $\gamma_1, \gamma_2$  and  $\gamma_3$  are real,  $\gamma_3 > 0$ ,  $\gamma_1 + \gamma_2 + \gamma_3 + N > 1$  and  $(t_1 - t_3)(t_2 - t_3) < 0$ . Then the equation  $c_{N+1}(q) = 0$  has all its roots real and unequal.*

*Proof.* We will show that the polynomial  $c_r(q)$  ( $1 \leq r \leq N$ ) has  $r$  real distinct roots  $s_i^{(r)}$  ( $i = 1, \dots, r$ ) such that

$$s_1^{(r)} < s_1^{(r-1)} < s_2^{(r)} < s_2^{(r-1)} < \dots < s_{r-1}^{(r)} < s_{r-1}^{(r-1)} < s_r^{(r)}$$

by induction on  $r$ . The case  $r = 1$  is trivial. Let  $k \in \mathbb{N}$  and assume that the statement is true for  $r \leq k$ . From the assumption of the induction,

$$s_1^{(k)} < s_1^{(k-1)} < s_2^{(k)} < s_2^{(k-1)} < \dots < s_{k-1}^{(k-1)} < s_k^{(k)}.$$

It follows from Proposition 2.2 (i) that the sign of the coefficient of  $q^r$  in  $c_r(q)$  is that of  $(-1)^r$ . Then

$$\lim_{q \rightarrow -\infty} c_{k-1}(q) = +\infty, \quad \lim_{q \rightarrow +\infty} c_{k-1}(q) = (-1)^{k-1} \infty.$$

For any  $1 \leq i \leq k-2$ , since  $c_{k-1}(s_i^{(k-1)}) = c_{k-1}(s_{i+1}^{(k-1)}) = 0$  and  $s_i^{(k-1)} < s_{i+1}^{(k-1)} < s_{i+1}^{(k)}$ , the sign of the value  $c_{k-1}(s_{i+1}^{(k)})$  is  $(-1)^i$ . Furthermore, it follows from  $s_1^{(k)} < s_1^{(k-1)}$  and  $s_k^{(k)} > s_{k-1}^{(k-1)}$  that  $c_{k-1}(s_1^{(k)}) > 0$  and the sign of the value of  $c_{k-1}(s_k^{(k)})$  is  $(-1)^{k-1}$ . In conclusion, the sign of the value  $c_{k-1}(s_i^{(k)})$  is  $(-1)^{i-1}$  for any  $1 \leq i \leq k$ . Then it follows from Proposition 2.2 (ii) that the sign of the value  $c_{k+1}(s_i^{(k)})$  is  $(-1)^i$  for any  $1 \leq i \leq k$ . Since

$$(2.4) \quad \lim_{q \rightarrow -\infty} c_{k+1}(q) = +\infty, \quad \lim_{q \rightarrow +\infty} c_{k+1}(q) = (-1)^{k+1} \infty,$$

it follows from the intermediate value theorem that the polynomial  $c_{k+1}(E)$  has  $k+1$  real distinct roots  $s_i^{(k+1)}$  ( $1 \leq i \leq k+1$ ) such that the inequality  $s_1^{(k+1)} < s_1^{(k)} < s_2^{(k+1)} < s_2^{(k)} < \dots < s_k^{(k)} < s_{k+1}^{(k+1)}$  is satisfied.  $\square$

If  $t_1 = t$ ,  $t_2 = 1$  and  $t_3 = 0$ , then equation (2.1) is the well-known *Heun's equation*, which is a standard form of the second order Fuchsian differential equation with four singularities on  $\mathbb{CP}^1$ . Clearly the condition  $(t_1 - t_3)(t_2 - t_3) < 0$  is equivalent to  $t < 0$ .

### 3. PROOF OF THEOREM 1.1

In this section, we will apply Theorem 2.3 to prove Theorem 1.1. Recall the Hamiltonian of the  $BC_1$  Inozemtsev model as mentioned in Section 1:

$$(3.1) \quad H := H^{(l_0, l_1, l_2, l_3)} := -\frac{d^2}{dz^2} + \sum_{k=0}^3 l_k(l_k + 1) \wp(z + \frac{\omega_k}{2}).$$



Note that the Hamiltonian is *unchanged* by replacing  $l_k$  to  $-l_k - 1$  ( $k = 0, 1, 2, 3$ ). Let  $f(z)$  be an eigenfunction of  $H$  with an eigenvalue  $E$ , i.e.

$$(3.2) \quad (H - E)f(z) = \left( -\frac{d^2}{dz^2} + \sum_{k=0}^3 l_k(l_k + 1)\wp(z + \frac{\omega_k}{2}) - E \right) f(z) = 0.$$

Set  $x = \wp(z)$ . Applying the formula

$$\wp(z + \frac{\omega_i}{2}) = e_i + \frac{(e_i - e_{i'}) (e_i - e_{i''})}{\wp(z) - e_i}, \text{ where } \{i, i', i''\} = \{1, 2, 3\},$$

it is easy to see that equation (3.2) is equivalent to

$$(3.3) \quad \left\{ \frac{d^2}{dx^2} + \frac{1}{2} \left( \frac{1}{x - e_1} + \frac{1}{x - e_2} + \frac{1}{x - e_3} \right) \frac{d}{dx} - \frac{1}{4 \prod_{j=1}^3 (x - e_j)} \right. \\ \left. \cdot \left( \tilde{C} + l_0(l_0 + 1)x + \sum_{i=1}^3 l_i(l_i + 1) \frac{(e_i - e_{i'}) (e_i - e_{i''})}{x - e_i} \right) \right\} \tilde{f}(x) = 0,$$

where  $\tilde{f}(\wp(z)) = f(z)$  and  $\tilde{C} = -E + \sum_{i=1}^3 l_i(l_i + 1)e_i$ . Note that  $e_1 + e_2 + e_3 = 0$ . It is easy to see that the Riemann scheme of equation (3.3) is

$$\left\{ \begin{array}{cccc} e_1 & e_2 & e_3 & \infty \\ \frac{-l_1}{2} & \frac{-l_2}{2} & \frac{-l_3}{2} & \frac{-l_0}{2} \\ \frac{l_1+1}{2} & \frac{l_2+1}{2} & \frac{l_3+1}{2} & \frac{l_0+1}{2} \end{array} \right\}.$$

Let  $\tilde{\alpha}_i = -l_i/2$  or  $(l_i + 1)/2$  for each  $i \in \{0, 1, 2, 3\}$ . Set

$$\Phi^{(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3)}(x) = \prod_{j=1}^3 (x - e_j)^{\tilde{\alpha}_j} \quad \text{and} \quad \tilde{f}(x) = \Phi^{(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3)}(x) f(x).$$

Then  $\tilde{f}(x)$  solves equation (3.3) is equivalent to that  $f(x)$  satisfies

$$(3.4) \quad \frac{d^2 f(x)}{dx^2} + \sum_{i=1}^3 \frac{2\tilde{\alpha}_i + \frac{1}{2}}{x - e_i} \frac{df(x)}{dx} + \left( \frac{(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 - \frac{l_0}{2})(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 + \frac{l_0+1}{2})x}{(x - e_1)(x - e_2)(x - e_3)} \right. \\ \left. + \frac{\frac{E}{4} - e_1(\tilde{\alpha}_2 + \tilde{\alpha}_3)^2 - e_2(\tilde{\alpha}_1 + \tilde{\alpha}_3)^2 - e_3(\tilde{\alpha}_1 + \tilde{\alpha}_2)^2}{(x - e_1)(x - e_2)(x - e_3)} \right) f(x) = 0.$$

This equation is in the form of equation (2.1) by setting

$$\alpha = \tilde{\alpha}_0 + \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3,$$

$$\beta = -\tilde{\alpha}_0 + \frac{1}{2} + \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3,$$

$$\gamma_i = 2\tilde{\alpha}_i + \frac{1}{2}, \quad t_i = e_i, \quad i = 1, 2, 3,$$

$$q = -\left( \frac{E}{4} - e_1(\tilde{\alpha}_2 + \tilde{\alpha}_3)^2 - e_2(\tilde{\alpha}_1 + \tilde{\alpha}_3)^2 - e_3(\tilde{\alpha}_1 + \tilde{\alpha}_2)^2 + e_3\alpha\beta \right).$$

We assume  $\gamma_3 = 2\tilde{\alpha}_3 + \frac{1}{2} > 0$  as before. It is well known that  $e_j = e_j(\tau) \in \mathbb{R}$  and  $e_1 > e_3 > e_2$  provided  $\tau \in i\mathbb{R}_{>0}$ , i.e.

$$(3.5) \quad (e_1 - e_3)(e_2 - e_3) < 0 \quad \text{if } \tau \in i\mathbb{R}_{>0}.$$



Write  $f(x) = \sum_{r=0}^{\infty} c_r(x - e_3)^r$  with  $c_0 = 1$ , then  $c_r$  is a polynomial in  $E$  of degree  $r$ . We denote it by  $c_r(E)$ . We set  $N = -\alpha = -\tilde{\alpha}_0 - \tilde{\alpha}_1 - \tilde{\alpha}_2 - \tilde{\alpha}_3$  and assume  $N \in \mathbb{Z}_{\geq 0}$ . Then the propositions in the previous section hold true for  $c_r(E)$ . In particular, it follows from Proposition 2.1 that if  $c_{N+1}(E) = 0$ , then the differential equation (3.3) has a "polynomial" solution  $\tilde{f}(x) = \Phi^{(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3)}(x)f(x)$  in the sense that  $f(x)$  is a polynomial of degree no more than  $N$ .

Let  $P_{\tilde{\alpha}_0, \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3}(E)$  be the monic polynomial obtained by normalising  $c_{N+1}(E)$ . Then

$$\deg P_{\tilde{\alpha}_0, \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3}(E) = N + 1 = -\tilde{\alpha}_0 - \tilde{\alpha}_1 - \tilde{\alpha}_2 - \tilde{\alpha}_3 + 1.$$

By Theorem 2.3 and (3.5), we immediately obtain the following theorem.

**Theorem 3.1.** *Assume that  $l_0, l_1, l_2, l_3 \in \mathbb{R}$ ,  $N = -\tilde{\alpha}_0 - \tilde{\alpha}_1 - \tilde{\alpha}_2 - \tilde{\alpha}_3 \in \mathbb{Z}_{\geq 0}$ ,  $2\tilde{\alpha}_3 + 1/2 > 0$ ,  $-\tilde{\alpha}_0 + \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 + 1/2 > 0$  and  $\tau \in i\mathbb{R}_{>0}$ . Then the equation  $P_{\tilde{\alpha}_0, \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3}(E) = 0$  has all its roots real and unequal.*

Remark that we do not need to assume  $l_j \in \mathbb{Z}$  in Theorem 3.1.

From now on, we assume that  $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$ . We recall the following important result from [17], which establishes the precise relation between the spectral polynomial  $Q^{(l_0, l_1, l_2, l_3)}(E)$  and the aforementioned polynomial  $P_{\tilde{\alpha}_0, \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3}(E)$ . This plays a key role in our proof of Theorem 1.1.

If  $l_0 + l_1 + l_2 + l_3$  is even, then the spectral polynomial  $Q(E) = Q^{(l_0, l_1, l_2, l_3)}(E)$  of  $H^{(l_0, l_1, l_2, l_3)}$  is written as  $Q(E) = P^{(0)}(E)P^{(1)}(E)P^{(2)}(E)P^{(3)}(E)$ , where

$$\begin{aligned} P^{(0)}(E) &= P_{-l_0/2, -l_1/2, -l_2/2, -l_3/2}(E), \\ P^{(1)}(E) &= \begin{cases} P_{-l_0/2, -l_1/2, (l_2+1)/2, (l_3+1)/2}(E), & l_0 + l_1 \geq l_2 + l_3 + 2, \\ 1, & l_0 + l_1 = l_2 + l_3, \\ P_{(l_0+1)/2, (l_1+1)/2, -l_2/2, -l_3/2}(E), & l_0 + l_1 \leq l_2 + l_3 - 2, \end{cases} \\ P^{(2)}(E) &= \begin{cases} P_{-l_0/2, (l_1+1)/2, -l_2/2, (l_3+1)/2}(E), & l_0 + l_2 \geq l_1 + l_3 + 2, \\ 1, & l_0 + l_2 = l_1 + l_3, \\ P_{(l_0+1)/2, -l_1/2, (l_2+1)/2, -l_3/2}(E), & l_0 + l_2 \leq l_1 + l_3 - 2, \end{cases} \\ P^{(3)}(E) &= \begin{cases} P_{-l_0/2, (l_1+1)/2, (l_2+1)/2, -l_3/2}(E), & l_0 + l_3 \geq l_1 + l_2 + 2, \\ 1, & l_0 + l_3 = l_1 + l_2, \\ P_{(l_0+1)/2, -l_1/2, -l_2/2, (l_3+1)/2}(E), & l_0 + l_3 \leq l_1 + l_2 - 2. \end{cases} \end{aligned}$$

If  $l_0 + l_1 + l_2 + l_3$  is odd, then the spectral polynomial  $Q(E)$  is written as  $Q(E) = P^{(0)}(E)P^{(1)}(E)P^{(2)}(E)P^{(3)}(E)$ , where

$$\begin{aligned} P^{(0)}(E) &= \begin{cases} P_{-l_0/2, (l_1+1)/2, (l_2+1)/2, (l_3+1)/2}(E), & l_0 \geq l_1 + l_2 + l_3 + 3, \\ 1, & l_0 = l_1 + l_2 + l_3 + 1, \\ P_{(l_0+1)/2, -l_1/2, -l_2/2, -l_3/2}(E), & l_0 \leq l_1 + l_2 + l_3 - 1, \end{cases} \\ P^{(1)}(E) &= \begin{cases} P_{(l_0+1)/2, -l_1/2, (l_2+1)/2, (l_3+1)/2}(E), & l_1 \geq l_0 + l_2 + l_3 + 3, \\ 1, & l_1 = l_0 + l_2 + l_3 + 1, \\ P_{-l_0/2, (l_1+1)/2, -l_2/2, -l_3/2}(E), & l_1 \leq l_0 + l_2 + l_3 - 1, \end{cases} \end{aligned}$$

$$P^{(2)}(E) = \begin{cases} P_{(l_0+1)/2, (l_1+1)/2, -l_2/2, (l_3+1)/2}(E), & l_2 \geq l_0 + l_1 + l_3 + 3, \\ 1, & l_2 = l_0 + l_1 + l_3 + 1, \\ P_{-l_0/2, -l_1/2, (l_2+1)/2, -l_3/2}(E), & l_2 \leq l_0 + l_1 + l_3 - 1, \end{cases}$$

$$P^{(3)}(E) = \begin{cases} P_{(l_0+1)/2, (l_1+1)/2, (l_2+1)/2, -l_3/2}(E), & l_3 \geq l_0 + l_1 + l_2 + 3, \\ 1, & l_3 = l_0 + l_1 + l_2 + 1, \\ P_{-l_0/2, -l_1/2, -l_2/2, (l_3+1)/2}(E), & l_3 \leq l_0 + l_1 + l_2 - 1. \end{cases}$$

Furthermore, it was shown in [17, Theorem 3.2] that the equations  $P^{(i)}(E) = 0$  and  $P^{(j)}(E) = 0$  ( $i \neq j$ ) do not have common solutions. We sketch the proof here for the reader's convenience.

*Proof.* Assume that there exists a common solution  $E = E_0$ . Let  $\tilde{f}^{(i)}(x)$  (resp.  $\tilde{f}^{(j)}(x)$ ) be the solution of equation (3.3) with  $E = E_0$ . Since  $\tilde{f}^{(i)}(\wp(z))$  and  $\tilde{f}^{(j)}(\wp(z))$  form a basis of solutions to (3.2), the Wronskian is a non-zero constant. However it contradicts that the periodicity of  $\tilde{f}^{(i)}(\wp(z))$  and  $\tilde{f}^{(j)}(\wp(z))$  with respect to the shift  $z \rightarrow z + \omega_1$  or  $z \rightarrow z + \omega_2$  is different.  $\square$

The following result proves Theorem 1.1 under the condition (i).

**Theorem 3.2.** *Assume that  $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$ . If  $l_3 = 0$ ,  $l_0 \geq l_1 + l_2 - 1$  and  $\tau \in i\mathbb{R}_{>0}$ , then the zeros of the spectral polynomial  $Q^{(l_0, l_1, l_2, l_3)}(E)$  are all real and unequal.*

*Proof.* We only need to show that the zeros of each polynomial  $P^{(j)}(E)$ ,  $j \in \{0, 1, 2, 3\}$ , are all real and unequal.

First we consider the case that  $l_0 + l_1 + l_2$  is even. Then  $l_0 \geq l_1 + l_2 - 1$  gives

$$(3.6) \quad l_0 \geq l_1 + l_2.$$

For  $P^{(0)}(E)$ , we have

$$(\tilde{\alpha}_0, \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3) = (-l_0/2, -l_1/2, -l_2/2, -l_3/2).$$

Since  $2\tilde{\alpha}_3 + 1/2 = -l_3 + 1/2 = 1/2 > 0$  and

$$-\tilde{\alpha}_0 + \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 + 1/2 = \frac{l_0 - l_1 - l_2 + 1}{2} > 0,$$

it follows from Theorem 3.1 that the zeros of  $P^{(0)}(E)$  are all real and unequal. Note from (3.6) and  $l_3 = 0$  that  $l_0 + l_1 \geq l_2 + l_3$ . Hence  $P^{(1)}(E) = P_{-l_0/2, -l_1/2, (l_2+1)/2, (l_3+1)/2}(E)$  or 1. If  $P^{(1)}(E) = P_{-l_0/2, -l_1/2, (l_2+1)/2, (l_3+1)/2}(E)$ , then  $2\tilde{\alpha}_3 + 1/2 = l_3 + 3/2 > 0$  and

$$-\tilde{\alpha}_0 + \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 + 1/2 = \frac{l_0 - l_1 + l_2 + 3}{2} > 0.$$

Again Theorem 3.1 implies that the zeros of  $P^{(1)}(E)$  are all real and unequal. It is shown similarly that the zeros of both  $P^{(2)}(E)$  and  $P^{(3)}(E)$  are all real and unequal.

We consider the remaining case that  $l_0 + l_1 + l_2$  is odd.

For  $P^{(0)}(E)$ , since  $l_0 \geq l_1 + l_2 - 1$ , we need to consider the cases  $l_0 = l_1 + l_2 - 1$  and  $l_0 \geq l_1 + l_2 + 1$  separately. If  $l_0 = l_1 + l_2 - 1$ , then

$$P^{(0)}(E) = P_{(l_0+1)/2, -l_1/2, -l_2/2, -l_3/2}(E) \quad \text{and so} \quad \deg P^{(0)}(E) = 1.$$

If  $l_0 \geq l_1 + l_2 + 1$ , then  $P^{(0)}(E) = P_{-l_0/2, (l_1+1)/2, (l_2+1)/2, (l_3+1)/2}(E)$  or 1. Again Theorem 3.1 implies that the zeros of  $P^{(0)}(E)$  are all real and unequal. Clearly  $l_0 \geq l_1 + l_2 - 1$  gives  $l_0 + l_2 + 1 \geq l_1 + 2l_2 \geq l_1$  and  $l_0 + l_1 + 1 \geq l_2$ . Hence  $P^{(1)}(E) = P_{-l_0/2, (l_1+1)/2, -l_2/2, -l_3/2}(E)$  or 1. Again it follows from Theorem 3.1 that the zeros of  $P^{(1)}(E)$  are all real and unequal. It is shown similarly that the zeros of  $P^{(2)}(E)$  and  $P^{(3)}(E)$  are all real and unequal. The proof is complete.  $\square$

*Remark 3.3.* If  $l_3 = 0$  and  $l_0 \leq l_1 + l_2 - 2$ , then the spectral polynomial  $Q(E)$  may have non-real zeros in the case  $\tau \in i\mathbb{R}_{>0}$ . For example, in the case  $l_0 = l_1 = l_2 = 2$  and  $l_3 = 0$ , the spectral polynomial  $Q(E)$  have non-real zeros in the case  $\tau \in i\mathbb{R}_{>0}$  where  $e_1 > e_3 > e_2$ . In fact, it was shown in [20, p. 396 and 400] that the spectral polynomial  $Q(E)$  in the case  $l_0 = l_1 = l_2 = 2$  and  $l_3 = 0$  coincides with that in the case  $l_0 = 3$  and  $l_1 = l_2 = l_3 = 1$  and

$$Q(E) = P^{(0)}(E)P^{(1)}(E)P^{(2)}(E)P^{(3)}(E), \quad P^{(k)}(E) = E - 15e_k, \quad (k = 1, 2, 3),$$

$$P^{(0)}(E) = E^4 - 54g_2E^2 - 864g_3E - 135g_2^2.$$

Then two roots of  $P^{(0)}(E) = 0$  are not real in the case  $e_1 > e_3 > e_2$ . We take the simplest case  $\tau = i$  for example, where  $g_3(i) = 0$  gives  $P^{(0)}(E) = E^4 - 54g_2E^2 - 135g_2^2$ , which clearly has two non-real roots because  $g_2(i) > 0$ .

Next, we need to apply some isomonodromic results in [21] to prove Theorem 1.1 under the assumptions (ii)-(iii). Recall  $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$ . If  $l_0 + l_1 + l_2 + l_3$  is even, we set

$$(3.7) \quad \begin{aligned} l_0^e &= (l_0 - l_1 - l_2 - l_3)/2 - 1, & l_1^e &= (l_0 - l_1 + l_2 + l_3)/2, \\ l_2^e &= (l_0 + l_1 - l_2 + l_3)/2, & l_3^e &= (l_0 + l_1 + l_2 - l_3)/2. \end{aligned}$$

Note that  $-1 - l_0^e = (-l_0 + l_1 + l_2 + l_3)/2$ . Then  $l_0^e, l_1^e, l_2^e, l_3^e \in \mathbb{Z}$  and

$$(3.8) \quad \begin{aligned} l_0^e + l_1^e + l_2^e + l_3^e &= 2l_0 - 1, & l_0^e + l_1^e - l_2^e - l_3^e &= -2l_1 - 1, \\ l_0^e - l_1^e + l_2^e - l_3^e &= -2l_2 - 1, & l_0^e - l_1^e - l_2^e + l_3^e &= -2l_3 - 1. \end{aligned}$$

By means of generalized Darboux transformations (i.e. for a suitable choice of  $(l'_0, l'_1, l'_2, l'_3)$ , there exists a differential operator  $L$  with coefficients being elliptic functions such that  $H^{(l'_0, l'_1, l'_2, l'_3)}L = LH^{(l_0, l_1, l_2, l_3)}$ ), it was proved in [21, Section 4] that the following eight operators are isomonodromic:

$$(3.9) \quad \begin{aligned} &H^{(l_0^e, l_1^e, l_2^e, l_3^e)}, H^{(l_1^e, l_0^e, l_3^e, l_2^e)}, H^{(l_2^e, l_3^e, l_0^e, l_1^e)}, H^{(l_3^e, l_2^e, l_1^e, l_0^e)}, \\ &H^{(l_0, l_1, l_2, l_3)}, H^{(l_1, l_0, l_3, l_2)}, H^{(l_2, l_3, l_0, l_1)}, H^{(l_3, l_2, l_1, l_0)}. \end{aligned}$$

Note that coincidence of the monodromy implies the coincidence of the spectral polynomial  $Q(E)$ 's, because the zeros of the spectral polynomial

is characterized by the double periodicity up to signs of the eigenfunction  $\tilde{f}(\wp(z))$  (see [17, Section 3]). Therefore we have

$$(3.10) \quad Q^{(l_0^e, l_1^e, l_2^e, l_3^e)}(E) = Q^{(l_1^e, l_0^e, l_3^e, l_2^e)}(E) = Q^{(l_2^e, l_3^e, l_0^e, l_1^e)}(E) = Q^{(l_3^e, l_2^e, l_1^e, l_0^e)}(E) \\ = Q^{(l_0, l_1, l_2, l_3)}(E) = Q^{(l_1, l_0, l_3, l_2)}(E) = Q^{(l_2, l_3, l_0, l_1)}(E) = Q^{(l_3, l_2, l_1, l_0)}(E).$$

If  $l_0 + l_1 + l_2 + l_3$  is odd, we set

$$(3.11) \quad l_0^o = (l_0 + l_1 + l_2 + l_3 + 1)/2, \quad l_1^o = (l_0 + l_1 - l_2 - l_3 - 1)/2, \\ l_2^o = (l_0 - l_1 + l_2 - l_3 - 1)/2, \quad l_3^o = (l_0 - l_1 - l_2 + l_3 - 1)/2.$$

Then  $l_0^o, l_1^o, l_2^o, l_3^o \in \mathbb{Z}$  and

$$(3.12) \quad l_0^o + l_1^o + l_2^o + l_3^o = 2l_0 - 1, \quad l_0^o + l_1^o - l_2^o - l_3^o = 2l_1 + 1, \\ l_0^o - l_1^o + l_2^o - l_3^o = 2l_2 + 1, \quad l_0^o - l_1^o - l_2^o + l_3^o = 2l_3 + 1.$$

It follows from [21, Section 4] that the following eight operators are isomonodromic.

$$(3.13) \quad H^{(l_0^o, l_1^o, l_2^o, l_3^o)}, H^{(l_1^o, l_0^o, l_3^o, l_2^o)}, H^{(l_2^o, l_3^o, l_0^o, l_1^o)}, H^{(l_3^o, l_2^o, l_1^o, l_0^o)}, \\ H^{(l_0, l_1, l_2, l_3)}, H^{(l_1, l_0, l_3, l_2)}, H^{(l_2, l_3, l_0, l_1)}, H^{(l_3, l_2, l_1, l_0)}.$$

Therefore we have

$$(3.14) \quad Q^{(l_0^o, l_1^o, l_2^o, l_3^o)}(E) = Q^{(l_1^o, l_0^o, l_3^o, l_2^o)}(E) = Q^{(l_2^o, l_3^o, l_0^o, l_1^o)}(E) = Q^{(l_3^o, l_2^o, l_1^o, l_0^o)}(E) \\ = Q^{(l_0, l_1, l_2, l_3)}(E) = Q^{(l_1, l_0, l_3, l_2)}(E) = Q^{(l_2, l_3, l_0, l_1)}(E) = Q^{(l_3, l_2, l_1, l_0)}(E).$$

We apply Theorem 3.2 to Eqs.(3.10, 3.14) and immediately obtain

**Proposition 3.4.** *Let  $l_3 = 0$ ,  $l_0, l_1, l_2 \in \mathbb{Z}_{\geq 0}$ ,  $l_0 \geq l_1 + l_2 - 1$  and  $\tau \in i\mathbb{R}_{>0}$ .*

- (i) *If  $l_0 + l_1 + l_2$  is even, then the zeros of the spectral polynomial  $Q^{(l_0^e, l_1^e, l_2^e, l_3^e)}(E)$  are all real and unequal, where  $l_0^e, l_1^e, l_2^e, l_3^e$  are defined by Eq.(3.7).*
- (ii) *If  $l_0 + l_1 + l_2$  is odd, then the zeros of the spectral polynomial  $Q^{(l_0^o, l_1^o, l_2^o, l_3^o)}(E)$  are all real and unequal, where  $l_0^o, l_1^o, l_2^o, l_3^o$  are defined by Eq.(3.11).*

In order to obtain the conditions (ii)-(iii) in Theorem 1.1, we need to investigate what conditions  $l_0^e, l_1^e, l_2^e, l_3^e$  and  $l_0^o, l_1^o, l_2^o, l_3^o$  satisfy in Proposition 3.4. We assume that  $l_3 = 0$ ,  $l_0, l_1, l_2 \in \mathbb{Z}_{\geq 0}$  and  $l_0 \geq l_1 + l_2 - 1$ .

If  $l_0 + l_1 + l_2$  is even, then  $l_0 \geq l_1 + l_2$  and it follows from  $l_3 = 0$  and (3.8) that  $l_0^e - l_1^e - l_2^e + l_3^e = -1$ , i.e.

$$(3.15) \quad l_0^e + l_3^e + 1 = l_1^e + l_2^e.$$

It follows from  $l_0 \geq 0$ ,  $l_1 \geq 0$ ,  $l_2 \geq 0$  and  $l_0 - l_1 - l_2 \geq 0$  that

$$(3.16) \quad l_0^e + l_1^e + l_2^e + l_3^e + 1 \geq 0, \quad l_0^e + l_1^e - l_2^e - l_3^e + 1 \leq 0, \\ l_0^e - l_1^e + l_2^e - l_3^e + 1 \leq 0, \quad l_0^e = (l_0 - l_1 - l_2)/2 - 1 \geq -1,$$

namely

$$(3.17) \quad l_0^e + l_1^e + l_2^e + l_3^e \geq -1, \quad l_2^e + l_3^e \geq l_0^e + l_1^e + 1, \\ l_1^e + l_3^e \geq l_0^e + l_2^e + 1, \quad l_0^e \geq -1.$$

Together with (3.15), we easily obtain

$$(3.18) \quad \begin{aligned} 2l_1^e + l_2^e &= l_1^e + l_0^e + l_3^e + 1 \geq 2l_0^e + l_2^e + 2, \\ 2l_2^e + l_1^e &= l_2^e + l_0^e + l_3^e + 1 \geq 2l_0^e + l_1^e + 2, \\ l_1^e + l_2^e + 2l_3^e &\geq 2l_0^e + l_1^e + l_2^e + 2, \end{aligned}$$

so

$$l_1^e \geq l_0^e + 1 \geq 0, \quad l_2^e \geq l_0^e + 1 \geq 0, \quad l_3^e \geq l_0^e + 1 \geq 0.$$

Recall  $l_0 \geq l_1 + l_2$ . If  $l_0 > l_1 + l_2 + 2$ , then (3.16) gives  $l_0^e > 0$  and so  $l_1^e, l_2^e, l_3^e > 0$ . If  $l_0 = l_1 + l_2 + 2$ , then  $l_0^e = 0$  and  $l_1^e, l_2^e, l_3^e > 0$ . If  $l_0 = l_1 + l_2$ , then  $l_0^e = -1$  and

$$l_1^e = \frac{l_0 - l_1 + l_2}{2} = l_2, \quad l_2^e = \frac{l_0 + l_1 - l_2}{2} = l_1, \quad l_3^e = \frac{l_0 + l_1 + l_2}{2} = l_0.$$

In this case,

$$H^{(l_0^e, l_1^e, l_2^e, l_3^e)} = H^{(-1, l_2, l_1, l_0)} = H^{(0, l_2, l_1, l_0)} = H^{(l_3, l_2, l_1, l_0)},$$

i.e. the operator  $H^{(l_0^e, l_1^e, l_2^e, l_3^e)}$  coincides with  $H^{(l_0, l_1, l_2, l_3)}$  by shifting the variable as  $x \rightarrow x - \omega_3/2$ .

Therefore, by rewriting  $l_0^e, l_1^e, l_2^e, l_3^e$  by  $l_0, l_1, l_2, l_3$  and ignoring the case  $l_0 = -1$ , we have the following result.

**Theorem 3.5.** *Assume that  $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$ ,  $l_0 + l_3 + 1 = l_1 + l_2$ ,  $l_2 + l_3 \geq l_0 + l_1 + 1$ ,  $l_1 + l_3 \geq l_0 + l_2 + 1$  and  $\tau \in i\mathbb{R}_{>0}$ . Then the zeros of the spectral polynomial  $Q^{(l_0, l_1, l_2, l_3)}(E)$  are all real and unequal.*

Now we turn to the case that  $l_0 + l_1 + l_2$  is odd. Then it follows from  $l_3 = 0$  and (3.12) that  $l_0^o - l_1^o - l_2^o + l_3^o = 1$ , i.e.

$$(3.19) \quad l_0^o + l_3^o = l_1^o + l_2^o + 1.$$

It follows from  $l_0 \geq 0, l_1 \geq 0, l_2 \geq 0$  and  $l_0 - l_1 - l_2 + 1 \geq 0$  that

$$(3.20) \quad \begin{aligned} l_0^o + l_1^o + l_2^o + l_3^o + 1 &\geq 0, \quad l_0^o + l_1^o - l_2^o - l_3^o - 1 \geq 0, \\ l_0^o - l_1^o + l_2^o - l_3^o - 1 &\geq 0, \quad l_3^o = (l_0 - l_1 - l_2 - 1)/2 \geq -1, \end{aligned}$$

namely

$$(3.21) \quad \begin{aligned} l_0^o + l_1^o + l_2^o + l_3^o &\geq -1, \quad l_0^o + l_1^o \geq l_2^o + l_3^o + 1, \\ l_0^o + l_2^o &\geq l_1^o + l_3^o + 1, \quad l_3^o \geq -1. \end{aligned}$$

Together with (3.19), we obtain

$$(3.22) \quad \begin{aligned} 2l_1^o + l_2^o &= l_1^o + l_0^o + l_3^o - 1 \geq l_2^o + 2l_3^o, \\ 2l_2^o + l_1^o &= l_2^o + l_0^o + l_3^o - 1 \geq 2l_3^o + l_1^o, \\ l_1^o + 2l_0^o + l_2^o &\geq l_1^o + l_2^o + 2l_3^o + 2, \end{aligned}$$

so

$$(3.23) \quad l_0^o \geq l_3^o + 1 \geq 0, \quad l_1^o \geq l_3^o \geq -1, \quad l_2^o \geq l_3^o \geq -1.$$

Recall  $l_0 \geq l_1 + l_2 - 1$ . If  $l_0 > l_1 + l_2 + 1$ , then (3.20) gives  $l_3^o > 0$  and so  $l_0^o, l_1^o, l_2^o > 0$ . If  $l_0 = l_1 + l_2 + 1$ , then  $l_3^o = 0 = l_3$  and  $l_0^o = (l_0 + l_1 + l_2 + 1)/2 = l_0$ ,  $l_1^o = (l_0 + l_1 - l_2 - 1)/2 = l_1$ ,  $l_2^o = (l_0 - l_1 + l_2 - 1)/2 = l_2$ , i.e.  $H^{(l_0^o, l_1^o, l_2^o, l_3^o)} = H^{(l_0, l_1, l_2, l_3)}$ . It remains to consider the case  $l_0 = l_1 + l_2 - 1$ . Then  $l_3^o = -1$  and  $l_0^o = l_0 + 1$ ,  $l_1^o = l_1 - 1$ ,  $l_2^o = l_2 - 1$ , i.e.  $H^{(l_0^o, l_1^o, l_2^o, l_3^o)} = H^{(l_0+1, l_1-1, l_2-1, -1)}$ . If  $l_1 > 0$  and  $l_2 > 0$ , then

$$H^{(l_0^o, l_1^o, l_2^o, l_3^o)} = H^{(l_0+1, l_1-1, l_2-1, 0)}$$

reduces to the case in Theorem 3.2; if  $l_1 = 0$ , then

$$H^{(l_0^o, l_1^o, l_2^o, l_3^o)} = H^{(l_0+1, -1, l_2-1, -1)} = H^{(l_2, 0, l_0, 0)}$$

reduces to the case in Theorem 3.2 too; if  $l_2 = 0$ , then

$$H^{(l_0^o, l_1^o, l_2^o, l_3^o)} = H^{(l_0+1, l_1-1, -1, -1)} = H^{(l_1, l_0, 0, 0)}$$

reduces to the case in Theorem 3.2 or Theorem B.

Therefore, by rewriting  $l_0^o, l_1^o, l_2^o, l_3^o$  by  $l_0, l_1, l_2, l_3$  and ignoring the case  $l_0 = l_1 + l_3 \pm 1$ , we have the following result.

**Theorem 3.6.** *Assume that  $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$ ,  $l_0 + l_3 = l_1 + l_2 + 1$ ,  $l_0 + l_1 \geq l_2 + l_3 + 1$ ,  $l_0 + l_2 \geq l_1 + l_3 + 1$  and  $\tau \in i\mathbb{R}_{>0}$ . Then the zeros of the spectral polynomial  $Q^{(l_0, l_1, l_2, l_3)}(E)$  are all real and unequal.*

We are in the position to prove Theorem 1.1.

*Proof of Theorem 1.1.* Theorem 1.1 follows directly from Theorems 3.2, 3.5 and 3.6.  $\square$

#### 4. FROM THE VIEWPOINT OF THE MONODROMY DATA

The purpose of this and next sections is to prove Theorem 1.2 from the viewpoint of the monodromy data of GLE (1.6). There are two ways to discuss the monodromy of GLE (1.6). One way is to project GLE (1.6) to a new equation on  $\mathbb{CP}^1$  via  $x = \wp(z)$ , which is a second order Fuchsian equation with five singular points  $\{e_1, e_2, e_3, \wp(p), \infty\}$  with  $\wp(p)$  being apparent. Since a vast literature has been devoted to studying second order linear ODEs defined on  $\mathbb{CP}^1$ , we could apply some known theories to this new ODE. However, the explicit formulas for the monodromy of this new ODE are not easy to compute. Therefore, it is more convenient for us to calculate the monodromy of GLE (1.6) on the torus  $E_\tau$  directly.

The monodromy representation of GLE (1.6) is a homomorphism  $\rho : \pi_1(E_\tau \setminus (E_\tau[2] \cup (\{\pm p\} + \Lambda_\tau)), q_0) \rightarrow SL(2, \mathbb{C})$ , where  $q_0 \notin E_\tau[2] \cup (\{\pm p\} + \Lambda_\tau)$  is a base point. Let  $\gamma_\pm \in \pi_1(E_\tau \setminus (E_\tau[2] \cup (\{\pm p\} + \Lambda_\tau)), q_0)$  be a simple loop encircling  $\pm p$  counterclockwise respectively, and  $\ell_j$ ,  $j = 1, 2$ , be two fundamental cycles of  $E_\tau$  connecting  $q_0$  with  $q_0 + \omega_j$  such that  $\ell_j$  does not intersect with  $L + \Lambda_\tau$  (here  $L$  is the straight segment connecting  $\pm p$ ) and satisfies

$$(4.1) \quad \gamma_+ \gamma_- = \ell_1 \ell_2 \ell_1^{-1} \ell_2^{-1} \text{ in } \pi_1(E_\tau \setminus (\{\pm p\} + \Lambda_\tau), q_0).$$

Since the local exponents of (1.6) at  $\pm p$  are  $-\frac{1}{2}$  and  $\frac{3}{2}$  and  $\pm p \notin E_\tau[2]$  are apparent singularities, we always have

$$(4.2) \quad \rho(\gamma_\pm) = -I_2.$$

For any  $k \in \{0, 1, 2, 3\}$ , the local exponents of GLE (1.6) at  $\omega_k/2$  are  $-l_k$  and  $l_k + 1$  with  $l_k \in \mathbb{N} \cup \{0\}$ . Since the potential  $I^{(l_0, l_1, l_2, l_3)}(z; p, \tau)$  is even elliptic, the local monodromy matrix of GLE (1.6) at  $\omega_k/2$  is  $I_2$  (see e.g. [16, Lemma 2.2]). Therefore, the monodromy group of GLE (1.6) is generated by  $\{-I_2, \rho(\ell_1), \rho(\ell_2)\}$ . Together with (4.1) and (4.2), we immediately obtain  $\rho(\ell_1)\rho(\ell_2) = \rho(\ell_2)\rho(\ell_1)$ , which implies that the monodromy group of GLE (1.6) is always *abelian* and hence *reducible*, i.e. *all the monodromy matrices have at least a common eigenfunction*. Clearly there are two cases:

- (a) Completely reducible, i.e. all the monodromy matrices have two linearly independent common eigenfunctions: Up to a common conjugation,  $\rho(\ell_1)$  and  $\rho(\ell_2)$  can be diagonalized as

$$\rho(\ell_1) = \begin{pmatrix} e^{-2\pi i s} & 0 \\ 0 & e^{2\pi i s} \end{pmatrix}, \quad \rho(\ell_2) = \begin{pmatrix} e^{2\pi i r} & 0 \\ 0 & e^{-2\pi i r} \end{pmatrix}$$

for some  $(r, s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ . Moreover, by using these two common eigenfunctions, we could express their Wronskian  $W$  in terms of  $(A, p)$  as  $W^2 = \mathcal{Q}^{(l_0, l_1, l_2, l_3)}(A; p, \tau)$ , where  $\mathcal{Q}^{(l_0, l_1, l_2, l_3)}(A; p, \tau)$  is precisely the polynomial introduced in Section 1; see [6]. So this case occurs if and only if  $\mathcal{Q}^{(l_0, l_1, l_2, l_3)}(A; p, \tau) \neq 0$ . This procedure is to use the second symmetric product of GLE (1.6) and has become well known in the literature; see [23, 16, 17] for example.

- (b) Not completely reducible, i.e. the space of common eigenfunctions is of dimension 1: Up to a common conjugation,

$$(4.3) \quad \rho(\ell_1) = \varepsilon_1 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \rho(\ell_2) = \varepsilon_2 \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix},$$

where  $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$  and  $C \in \mathbb{C} \cup \{\infty\}$ . Remark that if  $C = \infty$ , then (4.3) should be understood as

$$\rho(\ell_1) = \varepsilon_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(\ell_2) = \varepsilon_2 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

In this case,  $C$  is called the *monodromy data* of GLE (1.6). Clearly this case occurs if and only if  $\mathcal{Q}^{(l_0, l_1, l_2, l_3)}(A; p, \tau) = 0$ .

*Remark 4.1.* As discussed before, GLE (1.6) can be projected to a new Fuchsian ODE on  $\mathbb{CP}^1$ . Then the monodromy representation of this new ODE is irreducible if and only if Case (a) occurs, and reducible if and only if Case (b) occurs. Most of the references in the literature are devoted to the case of irreducible representation on  $\mathbb{CP}^1$ , but very few are devoted to studying reducible representation.



From now on, we consider the special case  $(l_0, l_1, l_2, l_3) = (1, 0, 0, 0)$  and denote  $\mathcal{Q}(A; p, \tau) = \mathcal{Q}^{(1,0,0,0)}(A; p, \tau)$  for convenience. Then GLE (1.6) becomes

$$(4.4) \quad y''(z) = \left[ \begin{array}{c} 2\wp(z) + \frac{3}{4}(\wp(z+p) + \wp(z-p)) \\ + A(\zeta(z+p) - \zeta(z-p)) + B \end{array} \right] y(z) \text{ in } E_\tau.$$

Now suppose  $\mathcal{Q}(A; p, \tau) = 0$ . Then as explained above, Case (b) occurs with some monodromy data  $C \in \mathbb{C} \cup \{\infty\}$ . The following result is to express  $\wp(p|\tau)$  in terms of this  $C$ .

**Theorem 4.A.** [5] Fix  $\tau \in \mathbb{H}$  and  $p \notin E_\tau[2]$ .

- (1) If the monodromy of GLE (4.4) is not completely reducible (i.e.  $\mathcal{Q}(A; p, \tau) = 0$ ), then the monodromy data  $C$  satisfies either

$$(4.5) \quad \wp(p|\tau) = \frac{-4(C\eta_1 - \eta_2)^3 - g_2(C\eta_1 - \eta_2)(C - \tau)^2 + 2g_3(C - \tau)^3}{(C - \tau)[12(C\eta_1 - \eta_2)^2 - g_2(C - \tau)^2]},$$

with  $(\varepsilon_1, \varepsilon_2) = (1, 1)$  or

$$(4.6) \quad \wp(p|\tau) = \frac{(\frac{g_2}{2} - 3e_k^2)(C\eta_1 - \eta_2) + \frac{g_2}{4}e_k(C - \tau)}{3e_k(C\eta_1 - \eta_2) + (\frac{g_2}{2} - 3e_k^2)(C - \tau)},$$

for some  $k \in \{1, 2, 3\}$  with

$$(4.7) \quad (\varepsilon_1, \varepsilon_2) = \begin{cases} (1, -1) & \text{if } k = 1, \\ (-1, 1) & \text{if } k = 2, \\ (-1, -1) & \text{if } k = 3. \end{cases}$$

Conversely, the following holds.

- (2-i) If  $C \in \mathbb{C} \cup \{\infty\}$  satisfies the cubic equation (4.5), then there exists  $A \in \mathbb{C}$  (and  $B$  is given by  $(p, A)$  via (1.7) with  $(l_0, l_1, l_2, l_3) = (1, 0, 0, 0)$ ) such that for the corresponding GLE (4.4), up to a common conjugation,

$$\rho(\ell_1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \rho(\ell_2) = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}.$$

- (2-ii) Fix  $k \in \{1, 2, 3\}$ . If  $C \in \mathbb{C} \cup \{\infty\}$  satisfies the equation (4.6), then there exists  $A \in \mathbb{C}$  such that for the corresponding GLE (4.4), up to a common conjugation,

$$\rho(\ell_1) = \varepsilon_1 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \rho(\ell_2) = \varepsilon_2 \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix},$$

where  $(\varepsilon_1, \varepsilon_2)$  is given by (4.7).

**Remark 4.2.** The formulas (4.5)-(4.6) first appeared in [16, (3.68)-(3.69)] without detailed proofs and later was obtained in [4] independently, as explicit expressions of Riccati type solutions of Painlevé VI equation. But their connection with the monodromy data seems not be well addressed. The assertion (1) can be proved directly without applying Painlevé VI equation. However, the converse part of the assertion (1), i.e. the assertion (2), is very delicate. Recall  $\deg_A \mathcal{Q} = 6$ . If  $\mathcal{Q}(A; p, \tau)$  has six distinct roots, then the

assertion (2) may follow from the assertion (1) and the 1-1 correspondence  $A \mapsto (\varepsilon_1, \varepsilon_2, C)$ . However, if  $\mathcal{Q}(A; p, \tau)$  has multiple roots, then it is not clear whether any  $C$  satisfying either (4.5) or (4.6) is the monodromy data of some GLE (4.4) or not. Without knowing the root structure of  $\mathcal{Q}(A; p, \tau)$ , the proof of the assertion (2) is to apply the connection between GLE (4.4) and Painlevé VI equation. Since the proof of Theorem 4.A is long and has nothing related to Theorem 1.2, we refer the proof to [5].

How to apply Theorem 4.A to obtain Theorem 1.2? For  $k \in \{1, 2, 3\}$ , we denote  $C_k$  to be the unique solution of equation (4.6). Then by Theorem 4.A, there exists  $A_k \in \mathbb{C}$  such that the monodromy representation of the corresponding GLE (4.4) is not completely reducible, i.e.  $\mathcal{Q}(A_k; p, \tau) = 0$ . Similarly, we let  $C_j, j \in \{4, 5, 6\}$  be the three solutions of the cubic equation (4.5). Again by Theorem 4.A, there exists  $A_j \in \mathbb{C}$  such that  $\mathcal{Q}(A_j; p, \tau) = 0$ . Clearly

$$A_k \neq A_j \text{ for any } k \in \{1, 2, 3\} \text{ and } j \in \{4, 5, 6\},$$

$$A_{k_1} \neq A_{k_2} \text{ for any } k_1 \neq k_2 \in \{1, 2, 3\},$$

because the  $(\varepsilon_1, \varepsilon_2)$ 's of the corresponding GLEs (4.4) are different. Consequently, once we can prove that the cubic equation (4.5) has three distinct solutions, i.e.

$$(4.8) \quad C_4 \neq C_5 \neq C_6 \neq C_4,$$

then we immediately obtain

$$A_k \neq A_j \text{ for any } k \neq j \in \{1, 2, 3, 4, 5, 6\},$$

which will give all the roots of  $\mathcal{Q}(A; p, \tau) = 0$  because  $\deg_A \mathcal{Q} = 6$ . Therefore,

**Corollary 4.3.** *Fix  $\tau \in \mathbb{H}$  and  $p \notin E_\tau[2]$ . Then  $\mathcal{Q}(A; p, \tau) = 0$  has six distinct roots if and only if the cubic equation (4.5) has three distinct solutions.*

In our case  $\tau \in i\mathbb{R}_{>0}$  and  $p \in (0, \frac{1}{2}) \cup (0, \frac{\tau}{2})$ , we just need to prove (4.8) and  $A_j \in \mathbb{R}$  for  $p \in (0, \frac{1}{2})$  (resp.  $A_j \in i\mathbb{R}$  for  $p \in (0, \frac{\tau}{2})$ ) for all  $j$  to obtain Theorem 1.2. The full details will be given in the next section.

## 5. PROOF OF THEOREM 1.2

This section is devoted to the proof of Theorem 1.2. First we prove the following result.

**Lemma 5.1.** *Let  $\tau \in i\mathbb{R}_{>0}$  and  $j \in \{1, 2, 3, 4, 5, 6\}$ . Then the followings hold.*

- (1) *If  $p \in (0, \frac{1}{2})$ , then  $C_j \in i\mathbb{R} \cup \{\infty\}$  if and only if  $A_j \in \mathbb{R}$ .*
- (2) *If  $p \in (0, \frac{\tau}{2})$ , then  $C_j \in i\mathbb{R} \cup \{\infty\}$  if and only if  $A_j \in i\mathbb{R}$ .*

*Proof.* Recall  $C_j$  is the monodromy data of the GLE (4.4):

$$(5.1) \quad y''(z) = I(z)y(z) \text{ in } E_\tau,$$

where

$$(5.2) \quad I(z) = \begin{bmatrix} 2\wp(z|\tau) + \frac{3}{4}(\wp(z+p|\tau) + \wp(z-p|\tau)) \\ +A_j(\zeta(z+p|\tau) - \zeta(z-p|\tau)) + B_j \end{bmatrix},$$

and  $B_j$  is given by  $(p, A_j)$  via (1.7) with  $(l_0, l_1, l_2, l_3) = (1, 0, 0, 0)$ .

(1) Since  $p \in (0, \frac{1}{2})$ , we have  $\bar{p} = p$ . Let

$$I_0(z) := \begin{bmatrix} 2\wp(z|\tau) + \frac{3}{4}(\wp(z+p|\tau) + \wp(z-p|\tau)) \\ +\bar{A}_j(\zeta(z+p|\tau) - \zeta(z-p|\tau)) + \bar{B}_j \end{bmatrix}.$$

Since  $\tau \in i\mathbb{R}_{>0}$ , we easily obtain  $\overline{I(\bar{z})} = I_0(z)$ . Consequently, if  $y(z)$  is a local solution of GLE (5.1)-(5.2) in a small domain bounded away from  $\pm p + \Lambda_\tau$ , then

$$(5.3) \quad \tilde{y}(z) := \overline{y(\bar{z})}$$

is a local solution of GLE  $y'' = I_0(z)y(z)$ . By (5.3), it is easy to prove that the monodromy representation of GLE  $y'' = I_0(z)y(z)$  has the same  $(\varepsilon_1, \varepsilon_2)$  as that of GLE (5.1)-(5.2) and the monodromy data is  $-\bar{C}_j$ .

If  $A_j \in \mathbb{R}$ , i.e.  $\bar{A}_j = A_j$ , then  $I_0(z) = I(z)$ , namely GLE  $y'' = I_0(z)y(z)$  coincides with GLE (5.1)-(5.2). Consequently, the monodromy data  $-\bar{C}_j = C_j$ , i.e.  $C_j \in i\mathbb{R} \cup \{\infty\}$ .

Conversely, if  $C_j \in i\mathbb{R} \cup \{\infty\}$ , we have  $-\bar{C}_j = C_j$ , namely GLE  $y'' = I_0(z)y(z)$  has the same monodromy group generators  $\rho(\ell_1), \rho(\ell_2)$  as GLE (5.1)-(5.2). Applying a uniqueness result of such GLE with respect to the monodromy group generators  $\rho(\ell_1), \rho(\ell_2)$  (see [6]), we conclude that  $I_0(z) = I(z)$ , which gives  $\bar{A}_j = A_j$ , i.e.  $A_j \in \mathbb{R}$ .

(2) Since  $p \in (0, \frac{\tau}{2})$ , i.e.  $p$  is purely imaginary, we have  $\bar{p} = -p$ . Define

$$\tilde{I}_0(z) := \begin{bmatrix} 2\wp(z|\tau) + \frac{3}{4}(\wp(z+p|\tau) + \wp(z-p|\tau)) \\ -\bar{A}_j(\zeta(z+p|\tau) - \zeta(z-p|\tau)) + \bar{B}_j \end{bmatrix}.$$

Then  $\overline{\tilde{I}(\bar{z})} = \tilde{I}_0(z)$ . The same argument as (1) implies that  $C_j \in i\mathbb{R} \cup \{\infty\}$  if and only if  $\tilde{I}_0(z) = I(z)$ , i.e.  $-\bar{A}_j = A_j$ , which is just  $A_j \in i\mathbb{R}$ .  $\square$

Next, we need to prove the following result.

**Theorem 5.2.** *Let  $\tau \in i\mathbb{R}_{>0}$  and  $p \in (0, \frac{1}{2}] \cup (0, \frac{\tau}{2}]$ . Then the three roots  $C$ 's of equation*

$$(5.4) \quad \wp(p|\tau) = \frac{-4(C\eta_1 - \eta_2)^3 - g_2(C\eta_1 - \eta_2)(C - \tau)^2 + 2g_3(C - \tau)^3}{(C - \tau)[12(C\eta_1 - \eta_2)^2 - g_2(C - \tau)^2]}$$

*are distinct and all belong to  $i\mathbb{R} \cup \{\infty\}$ .*

Before we go to prove Theorem 5.2, we are in the position to prove Theorem 1.2.

*Proof of Theorem 1.2.* Fix  $\tau \in i\mathbb{R}_{>0}$  and  $p \in (0, \frac{1}{2}) \cup (0, \frac{\tau}{2})$ . It is well known that

$$e_1, e_2, e_3, g_2, \eta_1, \wp(p) \in \mathbb{R} \text{ and } \eta_2 \in i\mathbb{R}.$$

Then it follows from equation (4.6) that  $C_k \in i\mathbb{R} \cup \{\infty\}$  for all  $k \in \{1, 2, 3\}$ . Together with the argument at the end of Section 4, it follows from Lemma 5.1 and Theorem 5.2 that  $A_j \in \mathbb{R}$  for  $p \in (0, \frac{1}{2})$  (resp.  $A_j \in i\mathbb{R}$  for  $p \in (0, \frac{\tau}{2})$ ) and are all distinct for  $j \in \{1, 2, 3, 4, 5, 6\}$ . Therefore,  $\mathcal{Q}(A; p, \tau) = 0$  has six distinct real roots for  $p \in (0, \frac{1}{2})$  and six distinct purely imaginary roots for  $p \in (0, \frac{\tau}{2})$ . Finally, by denoting  $y_j = A_j \wp'(p|\tau)$ , we conclude from  $\wp'(p|\tau) \in \mathbb{R}$  for  $p \in (0, \frac{1}{2})$  and  $\wp'(p|\tau) \in i\mathbb{R}$  for  $p \in (0, \frac{\tau}{2})$  that such  $y_j$ 's are real and give all the roots of  $\hat{\ell}_1(y; x, \tau) = 0$ , so  $\hat{\ell}_1(y; x, \tau) = 0$  has six distinct real roots. The proof is complete.  $\square$

We will exploit a conceptual idea to prove Theorem 5.2. Remark that  $C = \tau$  can not be a root of equation (5.4) because  $\wp(p|\tau) \neq \infty$ . Denote

$$X = \frac{C\eta_1(\tau) - \eta_2(\tau)}{\tau - C} \text{ and } x = \wp(p|\tau).$$

Then equation (5.4) is equivalent to

$$(5.5) \quad X^3 - 3xX^2 + \frac{g_2}{4}X + \frac{2g_3 + xg_2}{4} = 0.$$

Since  $\tau \in i\mathbb{R}_{>0}$ , it is well known that

$$g_2(\tau) > 0, \quad g_3(\tau) \in \mathbb{R}, \quad x = \wp(p|\tau) \begin{cases} \geq e_1(\tau) > 0 & \text{if } p \in (0, \frac{1}{2}], \\ \leq e_2(\tau) < 0 & \text{if } p \in (0, \frac{\tau}{2}]. \end{cases}$$

On the other hand, a straightforward computation implies that the discriminant of equation (5.5) is

$$\Delta = \varphi(x; \tau)/16,$$

where

$$\varphi(x; \tau) := 432g_2x^4 + 864g_3x^3 - 72g_2^2x^2 - 216g_2g_3x - g_2^3 - 108g_3^2.$$

**Lemma 5.3.** *Let  $\tau \in i\mathbb{R}_{>0}$  and  $p \in (0, \frac{1}{2}] \cup (0, \frac{\tau}{2}]$ . Then equation (5.5) has three real distinct roots  $X$ 's provided that one of the following conditions hold:*

- (1)  $p \in (0, \frac{1}{2}]$  and  $\tau = ib$  with  $b \geq 1$ .
- (2) for any fixed  $\tau \in i\mathbb{R}_{>0}$ , either  $p > 0$  is sufficiently small or  $\frac{1}{2} - p \geq 0$  is sufficiently small.
- (3)  $p \in (0, \frac{\tau}{2}]$  and  $\tau = ib$  with  $b \in (0, 1]$ .
- (4) for any fixed  $\tau \in i\mathbb{R}_{>0}$ ,  $p \in (0, \frac{\tau}{2}]$  satisfies either  $|p|$  or  $|\frac{\tau}{2} - p|$  is sufficiently small.

*Proof.* It is known that equation (5.5) has three real distinct roots if and only if the discriminant  $\Delta > 0$ , i.e.  $\varphi(x; \tau) > 0$ .

(1)-(2). Recall  $p \in (0, \frac{1}{2}]$  gives  $x \geq e_1$ . Since  $\tau = ib$  with  $b \geq 1$ , so  $e_1 > 0 \geq e_3 > e_2$ , i.e.  $g_3 = 4e_1e_2e_3 \geq 0$  and

$$0 < g_2 = 4(e_1^2 - e_2e_3) \leq 4e_1^2 \leq 4x^2.$$

Consequently,

$$\begin{aligned}\varphi(x; \tau) &= 72g_2x^2(6x^2 - g_2) + 216g_3x(4x^2 - g_2) - g_2^3 - 108g_3^2 \\ &\geq 72g_2 \times \frac{g_2}{4} \times \frac{g_2}{2} - g_2^3 - 108g_3^2 \\ &= 8g_2^3 - 108g_3^2 > 0,\end{aligned}$$

where  $g_2^3 - 27g_3^2 > 0$  is used in the last inequality. This proves (1).

To prove (2), we fix any  $\tau = ib$  with  $b \in (0, 1)$ . Then  $e_1 > e_3 > 0 > e_2$ . Denote

$$\alpha = e_1^2 \text{ and } \beta = -e_2e_3 = (e_1 + e_3)e_3 < 2\alpha.$$

Then  $g_2 = 4(\alpha + \beta)$  and  $e_1g_3 = -4\alpha\beta < 0$ . Consequently, a direct computation leads to

$$\begin{aligned}\varphi(e_1; \tau) &= 432g_2e_1^4 + 864g_3e_1^3 - 72g_2^2e_1^2 - 216g_2g_3e_1 - g_2^3 - 108g_3^2 \\ &= 64(2\alpha - \beta)^3 > 0.\end{aligned}$$

Therefore,  $\varphi(x; \tau) > 0$  if  $x - e_1(\tau) \geq 0$  is sufficiently small, namely provided that  $\frac{1}{2} - p \geq 0$  is sufficiently small. Finally, when  $p > 0$  is sufficiently small, then  $x = \wp(p|\tau)$  is sufficiently large, which clearly implies  $\varphi(x; \tau) > 0$ . This proves (2).

(3)-(4). Recall  $p \in (0, \frac{\tau}{2}]$  gives  $x \leq e_2 < 0$ . Since  $xg_3 \geq 0$ ,  $4x^2 \geq g_2$  for  $\tau = ib$  with  $b \in (0, 1]$  and  $-e_1e_3 < 2e_2^2$  for  $\tau = ib$  with  $b > 1$ , the proof is the same as that of (1)-(2) by exchanging the roles of  $e_1$  and  $e_2$ .  $\square$

**Lemma 5.4.** *Let  $\tau \in i\mathbb{R}_{>0}$  and  $p \in (0, \frac{1}{2}] \cup (0, \frac{\tau}{2}]$ . Then equation (5.5) has three real distinct roots  $X$ 's.*

*Proof.* First we consider  $p \in (0, \frac{1}{2}]$ . Instead of proving  $\varphi(x; \tau) > 0$  when  $\tau = ib$  and  $b \in (0, 1)$  (which seems non-trivial because  $g_3 < 0$ ), here we exploit a conceptual proof. Define

$$b_0 = \inf \left\{ b_1 > 0 \mid \begin{array}{l} \text{(5.5) has three real distinct roots} \\ \text{for } \tau = ib \text{ with } b > b_1 \text{ and } p \in (0, \frac{1}{2}] \end{array} \right\}.$$

Then  $b_0 \leq 1$ . We only need to prove  $b_0 = 0$ .

Suppose  $b_0 > 0$ . By the definition of  $b_0$  and Lemma 5.3, we have

- (i) for any  $p \in (0, \frac{1}{2}]$  and  $\tau = ib_0$ , equation (5.5) has three real roots  $X$ 's.
- (ii) there exists  $p_0 \in (0, \frac{1}{2})$  such that equation (5.5) with  $\tau = ib_0$  and  $p = p_0$  has a multiple root  $X_0$  with multiplicity  $m \in \{2, 3\}$ .

Now we fix  $\tau = ib_0$  and define

$$H(X; p) := X^3 - 3\wp(p|\tau_0)X^2 + \frac{g_2(\tau_0)}{4}X + \frac{2g_3(\tau_0) + \wp(p|\tau_0)g_2(\tau_0)}{4}.$$

Then  $H(X_0; p_0) = 0$ , so

$$(5.6) \quad \begin{aligned} H(X; p) &= \frac{1}{m!} \frac{\partial^m H}{\partial X^m}(X_0; p_0)(X - X_0)^m \\ &+ \frac{\partial H}{\partial p}(X_0; p_0)(p - p_0) + \text{higher order terms.} \end{aligned}$$

If  $\frac{\partial H}{\partial p}(X_0; p_0) \neq 0$ , then (5.6) implies that there exists  $p \in (0, 1/2)$  satisfying  $|p - p_0| > 0$  sufficiently small such that  $H(X; p) = 0$  has roots  $X$ 's in  $\mathbb{C} \setminus \mathbb{R}$ , a contradiction with (i). Thus  $\frac{\partial H}{\partial p}(X_0; p_0) = 0$ , i.e.

$$(5.7) \quad 12X_0^2 - g_2(\tau_0) = 0.$$

This together with  $H(X_0; p_0) = 0$  gives

$$(5.8) \quad 4X_0^3 + g_2(\tau_0)X_0 + 2g_3(\tau_0) = 0.$$

However, (5.7)-(5.8) leads to  $g_2(\tau_0)^3 - 27g_3(\tau_0)^2 = 0$ , a contradiction. Therefore,  $b_0 = 0$ , which completes the proof of the case  $p \in (0, \frac{1}{2}]$ .

The case  $p \in (0, \frac{\tau}{2}]$  can be proved in a similar way, and we omit the details here.  $\square$

We conclude this section by proving Theorem 5.2.

*Proof of Theorem 5.2.* Fix  $\tau \in i\mathbb{R}_{>0}$ . Then  $\eta_1(\tau) \in \mathbb{R}$  and  $\eta_2(\tau) \in i\mathbb{R}$ . So  $X = \frac{C\eta_1(\tau) - \eta_2(\tau)}{\tau - C} \in \mathbb{R}$  if and only if  $C \in i\mathbb{R} \cup \{\infty\} \setminus \{\tau\}$ . Therefore, Theorem 5.2 follows directly from Lemma 5.4.  $\square$

## REFERENCES

- [1] F. Beukers and A. Waall; *Lamé equations with algebraic solutions*. J. Differ. Equ. **197** (2004), 1-25.
- [2] F. Calogero and A. Degasperis; *Spectral transform and solitons*. Vol I. Lecture Notes in Computer Science, **144** (1982).
- [3] Z. Chen, T.J. Kuo and C.S. Lin; *Hamiltonian system for the elliptic form of Painlevé VI equation*. J. Math. Pures Appl. **106** (2016), 546-581.
- [4] Z. Chen, T.J. Kuo and C.S. Lin; *Simple zero property of some holomorphic functions on the moduli space of tori*. preprint, 2016.
- [5] Z. Chen, T.J. Kuo, C.S. Lin and K. Takemura; *On reducible monodromy representations of some generalized Lamé equation*. preprint, 2016.
- [6] Z. Chen, T.J. Kuo and C.S. Lin; *On generalized Lamé equations*. in preparation.
- [7] C. L. Chai, C. S. Lin and C. L. Wang; *Mean field equations, Hyperelliptic curves, and Modular forms: I*. Cambridge Journal of Mathematics **3** (2015), 127-274.
- [8] S. Dahmen; *Counting integral Lamé equations with finite monodromy by means of modular forms*. Master Thesis, Utrecht University 2003.
- [9] G. Darboux; *Sur une équation linéaire*. C. R. t. XCIV. 25 (1882), 1645-1648.
- [10] F. Gesztesy and R. Weikard; *Picard potentials and Hill's equation on a torus*. Acta Math. **176** (1996), 73-107.
- [11] D. Guzzetti; *The elliptic representation of the general Painlevé VI equation*. Comm. Pure Appl. Math. **55** (2002), 1280-1363.
- [12] G.H. Halphen; *Traité des Fonctions Elliptique II*, 1888.
- [13] E. L. Ince; *Further investigations into the periodic Lamé equations*. Proc. R. Soc. Edinb. **60** (1940), 83-99.

- [14] A.O. Smirnov; *Elliptic solitons and Heun's equation*. CRM Proc. Lecture Notes, 32, Amer. Math. Soc., Providence (2002), 287-305.
- [15] E.G.C. Poole; *Introduction to the theory of linear differential equations*. Oxford University Press, 1936.
- [16] K. Takemura; *The Hermite-Krichever Ansatz for Fuchsian equations with applications to the sixth Painlevé equation and to finite gap potentials*. Math. Z. **263** (2009), 149-194.
- [17] K. Takemura; *The Heun equation and the Calogero-Moser-Sutherland system I: the Bethe Ansatz method*. Comm. Math. Phys. **235** (2003), 467-494.
- [18] K. Takemura; *The Heun equation and the Calogero-Moser-Sutherland system II: the perturbation and the algebraic solution*. Electron. J. Differential Equations. **2004** no. 15 (2004), 1-30.
- [19] K. Takemura; *The Heun equation and the Calogero-Moser-Sutherland system III: the finite gap property and the monodromy*. J. Nonlinear Math. Phys. **11** (2004), 21-46.
- [20] K. Takemura; *The Heun equation and the Calogero-Moser-Sutherland system IV: the Hermite-Krichever Ansatz*. Comm. Math. Phys. **258** (2005), 367-403.
- [21] K. Takemura; *The Heun equation and the Calogero-Moser-Sutherland system V: generalized Darboux transformations*. J. Nonlinear Math. Phys. **13** (2006), 584-611.
- [22] A. Treibich and J. L. Verdier; *Revetements exceptionnels et sommes de 4 nombres triangulaires*. Duke Math. J. **68** (1992), 217-236.
- [23] E.T. Whittaker and G.N. Watson; *A course of modern analysis, 4th edition*. Cambridge University Press, 1927.

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